International Journal of Theoretical Physics, Vol. 18, No. 3, 1979

On the Potential x^{2N} and the Correspondence Principle

Richard L. Liboff

Schools of Electrical Engineering and Applied Physics, Cornell University, Ithaca, New York, 14853

Received March 1, 1979

Eigenenergies and frequencies are obtained for a particle oscillating in the potential $(1/2)k^N x^{2N}$, where k is a constant, x is displacement, and N is a real number. These potentials include the harmonic oscillator ($N = 1$) and the square well ($N = \infty$). The n^{th} eigenenergy has the form $A_N n^{\lambda(N)}$, where $\lambda(N) = 2N/(N + 1)$, and A_N is independent of n. Application is made to the correspondence principle for the potentials $N > 1$ and it is concluded the classical continuum is not obtained in Bohr's limit $n \to \infty$. Complete correspondence is attained in Planck's limit $h \to 0$, so that for these configurations the limits $h \to 0$ and $n \to \infty$ are not equivalent. A classical analysis of these potentials is included which reveals the relation $\log_E (\nu/\nu_N) = (N - 1)/2N$ between frequency ν and energy E, where the constant ν_N is independent of E.

1. INTRODUCTION

In this paper we consider an infinite class of potentials which in one limit includes the parabolic potential of the harmonic oscillator and in another limit, the potential of the infinite square well. The validity of the Sommerfeld quantization rules to these configurations is argued and then applied to construct the eigenenergies of the potentials considered. Application of these results is made in evaluation of the frequency spectra of the various potentials. Save for the case of the harmonic oscillator, all frequency spectra remain removed from the zero frequency line by a finite interval which is independent of quantum number. Thus, the classical spectrum is not obtained in Bohr's limit $n \to \infty$. However, all spectra obtained are seen to collapse uniformly to the classical continuum including zero frequency in Planck's limit, $h \to 0$. Thus for the infinite class of potentials considered, excluding the harmonic oscillator, the limits $n \to \infty$ and $h \to 0$ are not equivalent.

0020-7748/79/0300-0185$03.00/0

2. ANALYSIS

2.1. The Quantum Energy Spectrum. We wish to examine the energy spectrum of a particle trapped in the potential well

$$V_N(x) = \frac{k^N}{2} x^{2N} \tag{1}$$

where k is an effective spring constant and N is a finite numerical constant. This potential is of interest to physics because of the following property. Namely, since $V_N(k^{-1/2}) = 1/2$ for all N, it is evident that $V_N(x)$ approaches a square well with increasing N. On the other hand, for $N = 1$, the potential $V_1(x)$ is the parabola appropriate to the harmonic oscillator.

Both of these configurations have well-known solutions in quantum mechanics (Schiff, 1968). For the harmonic oscillator, eigenenergies are

$$\begin{aligned} E_n^{(1)} &= \hbar\omega(n + \tfrac{1}{2}) \qquad N = 1 \\ \omega^2 &\equiv k/m \end{aligned} \tag{2}$$

whereas eigenenergies for the one-dimensional box are given by

$$\begin{aligned} E_n^{(\infty)} &= n^2 E_1 \qquad\qquad N = \infty \\ E_1 &= h^2/8mL^2, \qquad L \equiv 2/k^{1/2} \end{aligned} \tag{3}$$

So for $N = 1$, $E_n \propto n$ and for $N = \infty$, $E_n \propto n^2$. It is natural then to conjecture that for intermediate N, namely, $1 < N < \infty$, eigenenergies are given by

$$E_n^{(N)} \propto \tilde{n}^{\lambda(N)} \qquad 1 < \lambda(N) < 2 \tag{4}$$

with $\lambda(N)$ monotonically increasing with N and $\tilde{n}$, where

$$\tilde{n} \equiv n + \tfrac{1}{2}$$

The validity of this conjecture is established within the accuracy of the Sommerfeld quantization rules.

2.2. The Action Integral. The Sommerfeld quantization rules (Sommerfeld, 1915) stipulate that the action variable

$$J = \oint p\, dx \tag{5}$$

is restricted to discrete values, which for a particle in a well with penetrable walls is given by

$$J = h(n + \tfrac{1}{2}) \equiv h\tilde{n} \tag{6}$$

where n is an integer. For the infinite square well ($N = \infty$) the correct condition is $J = nh$. In (5), the coordinate x is conjugate to the momentum p and the integral is over a cycle of the periodic motion.

The quantization condition (6) stems from a WKB analysis (Landau and Lifshitz, 1958) wherein it is found that this quantization condition is valid in the classical limit. Equivalently, it is valid when wavefunction wavelength is short compared to potential scale length. In that this wavelength diminishes with increasing eigenenergy, the quantization condition (6) will be valid for all eigenstates providing it is valid for the ground state. Since this is the case for the extreme potentials of the harmonic oscillator ($N = 1$), and the square well ($N = \infty$) and furthermore the potential (1) grows flatter at the base with increasing N, it is reasonable to assume that the energies stemming from (6) are a very good approximation for all N and any n, growing still more accurate with increasing n.

In order to validate the conjecture (4), we must evaluate the integral (5). From this result we may obtain the energy as a function of J or, with (6), as a function of n. From the expression for the energy of the particle we obtain

$$J_N = \oint dx[m(2E - k^N x^{2N})]^{1/2}$$

With the substitution

$$xk^{1/2} = (2E)^{1/2N} \sin \theta \tag{7}$$

and setting $\omega^2 \equiv k/m$, the preceding integral becomes

$$J_N = \omega^{-1}(2E)^{(N+1)/2N} \oint d\theta \cos \theta (1 - \sin^{2N} \theta)^{1/2}$$

Equivalently we may write

$$J_N = 2\pi\omega^{-1} E^{(N+1)/2N} G_N \tag{8}$$

where G_N is the pure number

$$G_N \equiv 2^{(N+1)/2N} \oint \frac{d\theta}{2\pi} \cos \theta (1 - \sin^{2N} \theta)^{1/2} \tag{9}$$

For example, for $N = 1$, $G_1 = 1$ and we obtain the well-known result appropriate to the harmonic oscillator (Goldstein, 1959), $J_1 = (2\pi/\omega)E$.

Inverting (8) gives

$$E = (\omega/2\pi G_N)^{2N/(N+1)} J_N^{2N/(N+1)} \tag{10}$$

With the quantization rule (6), we obtain

$$E_n = A_N(n + \tfrac{1}{2})^{2N/(N+1)} \equiv A_N \tilde{n}^{2N/(N+1)} \tag{11}$$

where we have written

$$A_N \equiv (\hbar\omega/G_N)^{2N/(N+1)} \tag{12}$$

From (11) we may conclude that

$$E_n \propto \tilde{n}^{\lambda(N)} \tag{13}$$

where

$$\lambda(N) = \frac{2N}{N+1} \tag{14}$$

This result agrees with a previous calculation of ter Haar (1964). For $N = 1$, (14) gives $\lambda = 1$, in agreement with the harmonic oscillator result (2). For $N = \infty$, (14) gives $\lambda = 2$ in agreement with (3) for the square-well configuration. For intermediate values $1 < N < \infty$ we see that $\lambda(N)$ is a monotonically increasing function of N which satisfies the second equation in (4). This observation together with the result (13) establishes the validity of the conjecture (4), to within the Sommerfeld quantization approximation (6).

2.3. The Quantum Frequency Spectra

The eigenenergies of a particle in the potential well (1) are given by (11),

$$E_n = A_N \tilde{n}^{2N/(N+1)}$$

where A_N, as given by (12), is independent of quantum number n. The frequencies of emission ν for this configuration are given by the Bohr rule

$$h\nu = E_{n+\Delta} - E_n \tag{15}$$

where by selection rules, Δ is an odd number. This latter equation may be rewritten

$$\nu = \frac{A}{h}[(\tilde{n} + \Delta)^\lambda - \tilde{n}^\lambda] \tag{16}$$

The question we wish to entertain at this point is, what is the minimum quantum frequency which may be emitted by this system? Suppose $\tilde{n}$ is a continuous parameter. Then differentiation of (16) yields

$$\frac{d\nu}{d\tilde{n}} = \frac{A\lambda}{h}[(\tilde{n} + \Delta)^{\lambda-1} - \tilde{n}^{\lambda-1}] > 0$$

The inequality follows from the fact that $\lambda - 1 > 0$, so that ν, as given by (16), is an increasing function of n. We may conclude that the smallest value ν can have is given by the first allowed decay to the ground state:

$$\nu_{\text{Min}} = \frac{A}{h}\left[\left(1 + \frac{1}{2}\right)^\lambda - \left(\frac{1}{2}\right)^\lambda\right] > \frac{A}{h}$$

This inequality again follows from the fact that $\lambda > 1$ so that $(1 + \frac{1}{2})^\lambda > 1 + \frac{1}{2}$, whereas $(\frac{1}{2})^\lambda < \frac{1}{2}$. Thus, the *minimum frequency* emitted by a particle trapped in the potential well $V_N(x)$ with $N > 1$ obeys the inequality

$$\nu_{\text{Min}} > \frac{A}{h} > 0 \tag{17}$$

Noncorrespondence. The significance of this result is that the emission spectrum (16) remains separated from the classical $\nu = 0$ value for all transitions. Therefore the classical spectrum which densely fills a domain about and includes the $\nu = 0$ value is not contained in the spectrum of the potential $V_N(x)$ for $N > 1$.

On the other hand in the high quantum number limit, we see from (16), with $\Delta = 1$,

$$\delta\nu \sim \frac{A}{h}\lambda(\lambda - 1)n^{\lambda-2} \tag{18}$$

It follows that for $\lambda < 2$, $\delta\nu \sim 0$ with increasing n, resulting in a continuous spectrum. The complete spectrum remains separated from the origin by $\nu_{\text{Min}} > 0$.

We may conclude that the quantum frequency spectrum (16) only partially coalesces with the classical spectrum in the limit of large quantum numbers. The potential configurations (1) then serve as counterexamples (Liboff, 1975) to the Bohr correspondence principle, which states that the classical frequency spectrum is obtained from the quantum spectrum in the limit of large quantum numbers (Bohr, 1914, 1920; van der Waerden, 1968; Jammer, 1966).

However, from (12) and (17) we find

$$\nu_{\text{Min}} \propto A/h \propto h^{(N-1)/(N+1)} \tag{19}$$

so that for $N > 1$, $\nu_{\text{Min}} \to 0$ with h. Furthermore, from (18), we see that the separation of frequency $\delta\nu$ also is proportional to A/h. We may conclude that the quantum spectrum (16) collapses to the classical continuum including the $\nu = 0$ value, in Planck's limit $h \to 0$ (Planck, 1906; Jammer, 1966). Furthermore, regarding the emission spectrum from a particle trapped in the potential well (1), with $N > 1$, we see that the limits $h \to 0$ and $n \to \infty$ are *not* equivalent.

2.4. The Classical Continuum

In this section we wish to establish that the classical frequency spectrum for the potential (1) comprises a continuum including the value $\nu = 0$. It suffices to show that ν varies continuously with energy and includes the value

$\nu = 0$. To these ends we recall Hamilton's equation of motion (Goldstein, 1959) for the angle $\int \nu dt$,

$$\frac{\partial E(J)}{\partial J} = \nu \tag{20}$$

The relationship between E and J was previously obtained and is given in (8). There results

$$\nu = \left(\frac{dJ}{dE}\right)^{-1} = \frac{\omega}{2\pi G_N}\frac{2N}{N+1}E^{(N-1)/2N} \equiv \nu_N E^{(N-1)/2N} \tag{21}$$

Only for the harmonic oscillator, $N = 1$, is frequency independent of energy. For all other N, the minimum frequency $\nu = 0$, is attained when $E = 0$. The frequency then increases with E, since $(N - 1)/2N > 0$. For $N = \infty$ one obtains

$$\nu \propto 1/E^{1/2}$$

which is appropriate to the particle in a square well or a rigid rotator.

The actual classical emission spectrum of a particle oscillating in the potential well (1), $N > 1$, is comprised of the fundamental (21) and subsequent harmonics. This property follows from the observation that the orbit $x(t)$ is symmetric about $x = 0$ and has period ν^{-1}. Therefore Fourier expansion of $x(t)$ contains the fundamental (21) and subsequent harmonics. These frequencies are dependent on E, or equivalently, initial conditions, and yield a continuous spectrum.

3. CONCLUSIONS

The eigenenergies and emission frequencies for the potential (1) have been obtained for all N, within the framework of the Sommerfeld quantization rules. Eigenenergies are given by (11)

$$E_n = A_n \tilde{n}^{\lambda(N)}, \qquad \lambda(N) \equiv \frac{2N}{N+1}$$

which for $N = 1$, reduces to the energy spectrum of the harmonic oscillator and for $N = \infty$, reduces to the energy spectrum of the box with rigid walls. The frequency spectrum is given by (16). In the limit of large quantum numbers this spectrum includes a continuum which is separated from the $\nu = 0$ value by $\nu_{\text{Min}} > 0$ as given by (17), and passing to the limit of large n does not secure the classical continuum. Correspondence is found to be obeyed, on the other hand, in Planck's limit, $h \to 0$. In this limit the total spectrum becomes continuous beginning at $\nu = 0$.

A relation is also obtained (21) for the energy dependence of the classical frequency for a particle in the potential (1). The form of this relation indicates that only for the harmonic oscillator ($N = 1$) is frequency independent of energy.

ACKNOWLEDGMENTS

Fruitful discussions with my colleagues K. Gottfried and T. Fine are gratefully acknowledged.

REFERENCES

Bohr, N. (1914). *Fysisk Tedsskrift*, **12**, 97; (1920). *Zeitschrift für Physik*, **2**, 423.

Goldstein, H. (1959). *Classical Mechanics*. Addison Wesley, Reading, Massachusetts.

Jammer, M. (1966). *The Conceptual Development of Quantum Mechanics*. McGraw-Hill, New York.

Landau, L., and Lifshitz, L. (1958). *Quantum Mechanics*. Addison Wesley, Reading, Massachusetts.

Liboff, R. L. (1975). *Foundations of Physics*, **5**, 271; see also, "On the Validity of the Bohr Correspondence Principle," to be published in, *Annals de la Fondation L. de Broglie*, Paris.

Planck, M. (1906). *Vorlesüngen über die Theorie der Wärmestrahlung*. Barth, Leipzig.

Schiff, L. I. (1968). *Quantum Mechanics*, 3rd ed. McGraw-Hill, New York.

Sommerfeld, A. (1915). *Münchener Berichte*, 425.

ter Haar, D. (1964). *Selected Problems in Quantum Mechanics*. Academic Press, New York.

van der Waerden, B. (1968). *Sources of Quantum Mechanics*. Dover, New York.